

Analysis of the Effect of a Mean Velocity Field on Mean Field Dynamo

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ABSTRACT

We study semi-analytically and in a consistent manner, the generation of a mean velocity field $\overline{\mathbf{U}}$ by helical MHD turbulence, and the effect that this field can have on a Mean Field Dynamo. Assuming a prescribed, maximally helical small scale velocity field, we show that large scale flows can be generated in MHD turbulent flows, via small scale Lorentz force. These flows back-react on the mean electromotive force of a Mean Field Dynamo through new terms, leaving the original α and β terms explicitly unmodified. Cross-helicity plays the key role in interconnecting all the effects. In the minimal τ closure that we chose to work with, the effects are stronger for large relaxation times.

Key words: Magnetic fields, (*magnetohydrodynamics*) MHD, turbulence.

1 INTRODUCTION

Magnetohydrodynamical (MHD) turbulence seems to be a major physical process to generate and maintain the magnetic fields observed in most of the structures of the Universe (Brandenburg & Subramanian 2005a; Zel'dovich et al. 1983). When addressing the problem of the generation of large scale magnetic fields by small scale turbulent flows, a model known as Mean Field Dynamo (MFD) is usually considered (Moffatt 1978). Despite its simplicity and lack of broad applicability, it proved to be a very useful tool in studying qualitatively conceptual issues of large scale magnetic field generation. The mechanism is based on decomposing the fields into large scale, or mean fields, $\overline{\mathbf{U}}$, $\overline{\mathbf{B}}$, $\overline{\mathbf{A}}$ and small scale, turbulent ones \mathbf{u} , \mathbf{b} , \mathbf{a} . These small scale fields have very small coherence length, but their intensities can be higher than the one of the mean fields. In this mechanism the evolution equation for $\overline{\mathbf{B}}$ is written $\partial\overline{\mathbf{B}}/\partial t = \nabla \times (\overline{\mathbf{U}} \times \overline{\mathbf{B}} + \overline{\mathcal{E}} - \eta\overline{\mathbf{J}})$, where $\overline{\mathbf{J}} = \nabla \times \overline{\mathbf{B}}$, η is the Ohmic resistivity and $\overline{\mathcal{E}} = \overline{\mathbf{u} \times \mathbf{b}}$ the turbulent electromotive force (t.e.m.f.)¹. The term $\overline{\mathbf{U}} \times \overline{\mathbf{B}}$ is usually disregarded in the studies of MFD as the focus of most of them is to understand the generation of large scale quantities due to small scale effects. If homogeneous and isotropic turbulence is considered, the t.e.m.f can be written as $\overline{\mathcal{E}} = \alpha\overline{\mathbf{B}} -$

$\beta\overline{\mathbf{J}}$, with $\alpha \simeq -(1/3)\tau_{corr}[\overline{\mathbf{u} \cdot (\nabla \times \mathbf{u})} - \overline{\mathbf{b} \cdot (\nabla \times \mathbf{b})}]$ and $\beta \simeq (1/3)\tau_{corr}\overline{u^2}$, τ_{corr} being a correlation time (Moffatt 1972; Rudiger 1974; Pouquet et al. 1976; Zel'dovich et al. 1983). The dependencies on $\overline{\mathbf{B}}$ and \mathbf{b} are due to the back-reaction of those induced fields on the dynamo (Moffatt 1978; Brandenburg & Subramanian 2005a). In the kinematically driven dynamo considered here, the features of the generated fields crucially depend on the helicity of the flows: helical flows are at the base of the mechanisms to generate large scale fields, while non-helical flows would only produce small scale fields. This separation, however, is somewhat artificial, as small scale fields are also produced by helical turbulence (Brandenburg & Subramanian 2005a).

In this paper we want to address an issue not (or very seldom) considered in the literature up to now, namely, the induction of large scale flows $\overline{\mathbf{U}}$, also named shear flows, by the small scale turbulent fields, and how these induced flows back-react on the turbulent electromotive force $\overline{\mathcal{E}}$ of a MFD. On one side, we mean that the expression $\partial\overline{\mathbf{B}}/\partial t = \nabla \times (\overline{\mathcal{E}} - \eta\overline{\mathbf{J}})$ would be valid only during the time interval in which $\overline{\mathbf{U}} \times \overline{\mathbf{B}} \ll \overline{\mathcal{E}}$; and on the other, even if this conditions is satisfied, $\overline{\mathcal{E}}$ could be affected by the generation of $\overline{\mathbf{U}}$ and consequently its functional form should be modified to incorporate this effect. The generation of magnetic fields due to the action of these large scale velocity flows instead of by $\overline{\mathcal{E}}$ was recently addressed analytically by several authors (Rogachevskii & Kleeorin 2003, 2004; Rädler & Stepanov 2005), and was also studied numerically by Brandenburg (2001) and semi-analytically by Blackman & Brandenburg (2002). However, none of those

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¹ Overlines denote local spatial averages: they represent vector quantities whose intensities may vary in space, but whose direction and sense are uniform or vary smoothly. $\langle \rangle_{vol}$ denote volume averages, i.e., quantities that can depend only on time.

works addressed specifically the issue we want to analyze here.

We work in the framework of the two scale approximation, that consists in assuming that mean fields peak at a scale k_L^{-1} while turbulent ones do so at $k_S^{-1} \ll k_L^{-1}$, and also consider homogenous and isotropic turbulence. Although this kind of turbulence is of dubious validity when dealing with large scale fields, it serves well for initial, qualitative studies of the sought effects. Another assumption we shall make is that $\bar{\mathbf{B}}$ is force-free, i.e., of maximal current helicity. Although fields with this feature can be observed in certain astrophysical environments, they are not a generality, and also they are not seen in some numerical simulations. The main reason to use them here is to simplify the (heavy) mathematics, while maintaining a physically meaning scenario.

In order to find $\bar{\mathcal{E}}$ when Lorentz force acts on the plasma, we must solve a differential equation that contains terms with one point triple correlations, i.e., averages of products of three stochastic fields evaluated at the same point. This means that instead of dealing with only one equation to solve for $\bar{\mathcal{E}}$, we have to solve a hierarchy of them. In order to break this hierarchy and thus simplify the mathematical treatment of the problem we must choose a *closure prescription*, which consists in writing the high order correlations as functions of the lower order ones, but maintaining the physical features of the problem under study. In MHD the intensity of the non-linearities is measured by the magnetic Reynolds number, which is defined from the induction equation for the magnetic field, $\partial \mathbf{B} / \partial t = \nabla (\mathbf{U} \times \mathbf{B}) + \eta \nabla^2 \mathbf{B}$, as $R_m \sim (UB/l) / (\eta B/l^2) = Ul/\eta$, where now U is a characteristic velocity of the plasma, l is a characteristic length and η is the ohmic resistivity. Thus for $R_m \ll 1$ the non-linear terms can be neglected in front of the resistive ones, and therefore the equations become linear, while for $R_m \gg 1$ the resistive terms should be dropped off in front of the non-linear ones. The intermediate regime is more difficult to analyze. In this paper we shall consider $R_m \gg 1$, i.e., the non-linearities must be maintained, and consequently a closure scheme must be selected to deal with them. We choose to work with the so called minimal τ approximation (Blackman & Field 2002; Brandenburg & Subramanian 2005a), whereby the triple moments in the equation for $\bar{\mathcal{E}}$ will be considered as proportional to quadratic moments, and written in the form $\zeta \bar{\mathcal{E}}$, with the proportionality factor $\zeta \sim \tau_{rel}^{-1}$, where τ_{rel} is a relaxation time that can in principle be scale, and/or R_m , dependent. The validity of this closure was checked numerically (Brandenburg & Subramanian 2005b) for low Reynolds numbers, and was also verified for the case of passive scalar diffusion. We shall assume that it is also valid for all the triple correlations that may appear throughout this work. We consider boundary conditions such that all total divergencies vanish. These conditions may be a bit unrealistic for astrophysical systems, but they have two advantages: magnetic helicity becomes a gauge-invariant quantity (Berger & Field 1984), and the obtained results can be compared with numerical simulation, as to perform them it is customary to use those conditions. We consider fully helical, prescribed, \mathbf{u} fields.

Our starting points are the evolution equations for $\bar{\mathcal{E}}$ and for the magnetic helicities $H_{L,S}^M$, as the evolution of

these quantities is tightly interlinked (Blackman & Field 2002; Brandenburg & Subramanian 2005a) in the absence of shear flows. When including $\bar{\mathbf{U}}$ new terms appear in the equation for $\bar{\mathcal{E}}$, but the ones that drive the evolution of $\bar{\mathcal{E}}$ in absence of $\bar{\mathbf{U}}$, i.e., the α and β terms (Moffatt 1972; Rudiger 1974; Pouquet et al. 1976; Blackman & Field 2002), are not explicitly modified. For those new terms, further equations must be derived, that in turn show the subtleties of the interplay among $\bar{\mathbf{U}}$, \mathbf{u} , \mathbf{b} and $\bar{\mathbf{B}}$. Due to the chosen boundary conditions, the evolution equations for H_L^M and H_S^M do not explicitly depend on $\bar{\mathbf{U}}$, they will do so implicitly through $\bar{\mathcal{E}}$. We consider fully helical $\bar{\mathbf{U}}$ fields, so we study their growth through its associated kinetic helicity, $H^{\bar{\mathbf{U}}} \equiv \langle (\nabla \times \bar{\mathbf{U}}) \cdot \bar{\mathbf{U}} \rangle_{vol}$ and show that, for fully helical, prescribed \mathbf{u} , large scale flows will be always generated, as long as small scale Lorentz force is not null, i.e., if $(\nabla \times \mathbf{b}) \times \mathbf{b} \neq 0$. We shall consider two values for the magnetic Reynolds number, $R_m = 200$ and 2000 , and for each case analyze the effect of short and large τ_{rel} . In general we find that for short τ_{rel} (large ζ), i.e., strong non-linearities, the effect of large scale flow is negligible, thus producing results that practically do not differ from the ones in the absence of large scale flows. For large τ_{rel} (small ζ) the general effect is an enhancement of the electromotive force and the inverse cascade of magnetic helicity, this enhancement being stronger for $R_m = 2000$ than for $R_m = 200$.

2 MAIN EQUATIONS

Ohm's law for an electrically conducting fluid reads $\mathbf{E} = -\mathbf{U} \times \mathbf{B} + \eta \mathbf{J}$, with η the electric resistivity and \mathbf{J} the electric current. The equation for \mathbf{B} is the induction equation:

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E}, \quad (1)$$

and from $\mathbf{B} = \nabla \times \mathbf{A}$ we have

$$\mathbf{E} = -\nabla \Phi - \frac{\partial \mathbf{A}}{\partial t}, \quad (2)$$

which is an evolution equation for \mathbf{A} . The equation for the velocity field \mathbf{U} is the Navier-Stokes equation that, when considering only Lorentz force, reads

$$\frac{\partial \mathbf{U}}{\partial t} = -(\mathbf{U} \cdot \nabla) \mathbf{U} - \frac{\nabla p}{\rho} + (\nabla \times \mathbf{B}) \times \mathbf{B} - \nu \nabla \times (\nabla \times \mathbf{U}). \quad (3)$$

with ν the kinetic viscosity. To work within mean field theory (Moffatt 1978) we decompose the different fields as $\mathbf{B} = \bar{\mathbf{B}} + \mathbf{b}$, $\mathbf{A} = \bar{\mathbf{A}} + \mathbf{a}$, $\mathbf{U} = \bar{\mathbf{U}} + \mathbf{u}$, $\mathbf{E} = \bar{\mathbf{E}} + \mathbf{e}$ and $\Phi = \bar{\Phi} + \phi$, where any mean value of stochastic quantities vanishes. The derivation of the evolution equations for the mean and stochastic fields is a standard procedure, already described in the literature (Zel'dovich et al. 1983; Blackman & Field 2002). Consequently we only write here the results. Assuming incompressibility of the large and small scale flows, considering that $\bar{\mathbf{B}}$ is force free and working with Coulomb gauge for the vector potential, i.e., $\nabla \cdot \bar{\mathbf{A}} = 0 = \nabla \cdot \mathbf{a}$, we obtain the following equations for the mean fields

$$\frac{\partial \bar{\mathbf{B}}}{\partial t} = \nabla \times [\bar{\mathbf{U}} \times \bar{\mathbf{B}} + \bar{\mathcal{E}} - \eta \nabla \times \bar{\mathbf{B}}], \quad (4)$$

$$\frac{\partial \bar{\mathbf{A}}}{\partial t} = \bar{\mathbf{P}} [\bar{\mathbf{U}} \times \bar{\mathbf{B}} + \bar{\mathcal{E}}] - \eta \nabla \times \bar{\mathbf{B}}, \quad (5)$$

where $\bar{\mathcal{E}} = \overline{\mathbf{u} \times \mathbf{b}}$ is the t.e.m.f., and

$$\begin{aligned} \frac{\partial \bar{\mathbf{U}}}{\partial t} &= \bar{\mathbf{P}} \left[-(\bar{\mathbf{U}} \cdot \nabla) \bar{\mathbf{U}} - \overline{(\mathbf{u} \cdot \nabla) \mathbf{u}} + \overline{(\mathbf{b} \cdot \nabla) \mathbf{b}} \right] \\ &+ \nu \nabla^2 \bar{\mathbf{U}}. \end{aligned} \quad (6)$$

$(\bar{\mathbf{P}})_{ij} = \delta_{ij} \partial^2 - \partial_i \partial_j$ is the projector that selects the subspace of solutions of eq. (2) that satisfy the Coulomb gauge condition and the subspace of solutions of (3) that satisfy the incompressibility condition. Observe that eq. (6) shows that a large scale velocity field can be induced from an initially zero value, as long as $-\overline{(\mathbf{u} \cdot \nabla) \mathbf{u}} + \overline{(\mathbf{b} \cdot \nabla) \mathbf{b}} \neq 0$. The equations for the small scale fields read

$$\frac{\partial \mathbf{b}}{\partial t} = \nabla \times [\bar{\mathbf{U}} \times \mathbf{b} + \mathbf{u} \times \bar{\mathbf{B}} + \mathbf{u} \times \mathbf{b} - \bar{\mathcal{E}}] + \eta \nabla^2 \mathbf{b}, \quad (7)$$

$$\frac{\partial \mathbf{a}}{\partial t} = \bar{\mathbf{P}} [\bar{\mathbf{U}} \times \mathbf{b} + \mathbf{u} \times \bar{\mathbf{B}} + \mathbf{u} \times \mathbf{b} - \bar{\mathcal{E}}] + \eta \nabla^2 \mathbf{a}, \quad (8)$$

and

$$\begin{aligned} \frac{\partial \mathbf{u}}{\partial t} &= \bar{\mathbf{P}} \left[-(\bar{\mathbf{U}} \cdot \nabla) \mathbf{u} - (\mathbf{u} \cdot \nabla) \bar{\mathbf{U}} - (\mathbf{u} \cdot \nabla) \mathbf{u} + \overline{(\mathbf{u} \cdot \nabla) \mathbf{u}} \right. \\ &+ \left. (\bar{\mathbf{B}} \cdot \nabla) \mathbf{b} + (\mathbf{b} \cdot \nabla) \bar{\mathbf{B}} + (\mathbf{b} \cdot \nabla) \mathbf{b} - \overline{(\mathbf{b} \cdot \nabla) \mathbf{b}} \right] \\ &+ \nu \nabla^2 \mathbf{u}. \end{aligned} \quad (9)$$

2.1 Evolution Equation for Derived Quantities: Magnetic Helicity, Large Scale Kinetic Helicities, and the Stochastic Electromotive Force

As stated in the Introduction, we want to study if an initially zero, or very weak $\bar{\mathbf{U}}$, can grow due the action of a MFD, and back-react on it and on the magnetic helicity. This last quantity is defined as the average over the entire volume of the dot product $\mathbf{A} \cdot \mathbf{B}$ (Biskamp 1997). In this way, we write the magnetic helicity associated to the large and small scale fields respectively as $H_L^M \equiv \langle \bar{\mathbf{A}} \cdot \bar{\mathbf{B}} \rangle_{vol}$ and $H_S^M \equiv \langle \mathbf{a} \cdot \mathbf{b} \rangle_{vol}$, and by definition they can only depend on time. The evolution equations for H_L^M and H_S^M , for the chosen boundary conditions, read (Blackman & Field 2002)

$$\frac{\partial H_L^M}{\partial t} = 2 \langle \bar{\mathcal{E}} \cdot \bar{\mathbf{B}} \rangle_{vol} - 2\eta \langle (\nabla \times \bar{\mathbf{B}}) \cdot \bar{\mathbf{B}} \rangle_{vol}, \quad (10)$$

and

$$\frac{\partial H_S^M}{\partial t} = -2 \langle \bar{\mathcal{E}} \cdot \bar{\mathbf{B}} \rangle_{vol} - 2\eta \langle (\nabla \times \mathbf{b}) \cdot \mathbf{b} \rangle_{vol}. \quad (11)$$

Observe that these equations have the same form as the ones obtained in the absence of large scale flows. This fact is due to the selected boundary conditions: magnetic helicity can be injected into the system through the boundaries by large scale flows. Thus, in the case under consideration here, these flows cannot explicitly transport magnetic helicity between the different scales, they will act implicitly through $\bar{\mathcal{E}}$.

From the definition of $\bar{\mathcal{E}}$ given above, the evolution equation for the t.e.m.f. is $\partial \bar{\mathcal{E}} / \partial t = (\partial \mathbf{u} / \partial t) \times \mathbf{b} + \mathbf{u} \times (\partial \mathbf{b} / \partial t)$. Proceeding in a similar form as in Refs. (Blackman & Field 2002; Kandus et al. 2006), it now reads

$$\begin{aligned} \frac{\partial \bar{\mathcal{E}}}{\partial t} &= \frac{1}{3} \nabla \cdot \bar{\mathcal{E}} \bar{\mathbf{U}} + \frac{2}{3} \overline{\mathbf{u} \cdot \mathbf{b}} (\nabla \times \bar{\mathbf{U}}) \\ &+ \frac{1}{3} [(\nabla \times \mathbf{b}) \cdot \mathbf{b} - \overline{\mathbf{u} \cdot (\nabla \times \mathbf{u})}] \bar{\mathbf{B}} \end{aligned} \quad (12)$$

$$- \frac{1}{3} \overline{u^2} (\nabla \times \bar{\mathbf{B}}) + \eta \overline{\mathbf{u} \times \nabla^2 \mathbf{b}} + \nu \overline{\nabla^2 \mathbf{u} \times \mathbf{b}} + \bar{\mathbf{T}}.$$

where $\bar{\mathbf{T}}$ are the triple correlations for which a closure must be applied. Observe that the presence of $\bar{\mathbf{U}}$ adds two new terms to the equation for $\bar{\mathcal{E}}$ but does not explicitly modify the ones found in the absence of those flows. The influence of $\bar{\mathbf{U}}$ on the terms proportional to $\bar{\mathbf{B}}$ will be through the dependence of those terms with the magnetic helicities (cf. Ref. Blackman & Field (2002)). To gain conceptual clearness we shall make further physical hypotheses on our systems, that will also help to simplify the mathematics. One of them is to consider that large scale flows $\bar{\mathbf{U}}$ are fully helical. This is consistent with the concept of mean field dynamo and with the chosen boundary conditions. Therefore to track the evolution of the large scale velocity flow, we shall study its associated kinetic helicity, defined as $H^{\bar{\mathbf{U}}} = \langle \bar{\mathbf{W}} \cdot \bar{\mathbf{U}} \rangle_{vol}$, where $\bar{\mathbf{W}} = \nabla \times \bar{\mathbf{U}}$ is the vorticity. The derivation of the equation for $H^{\bar{\mathbf{U}}}$ is explained in the Appendix, and the result is

$$\frac{\partial H^{\bar{\mathbf{U}}}}{\partial t} \simeq 2 \overline{(\nabla \times \mathbf{b}) \times \mathbf{b} \cdot \bar{\mathbf{W}}} + 2\nu \nabla^2 \bar{\mathbf{U}} \cdot \bar{\mathbf{W}}. \quad (13)$$

where the semi-equality is due to the fact that we are approximating volume average by a local spatial average. It is well known that kinetic helicity is not conserved in MHD (Biskamp 1997), so eq. (13) is not an essentially new result. However it serves to our purposes in showing that large scale helical flows can be induced by turbulent \mathbf{b} -fields, provided they are not force-free. At this point we make another supposition: we take $\nabla \cdot \bar{\mathcal{E}} = 0$ which, besides being consistent with the chosen boundary conditions, means that the induction of large scale magnetic fields maximal for $\bar{\mathbf{U}} = 0$ (cf. eq. 4). Observe that by imposing Coulomb gauge on eq. (5)² we obtain a further constraint on the mean fields, namely $\nabla \cdot \bar{\mathcal{E}} = -\nabla \cdot (\bar{\mathbf{U}} \times \bar{\mathbf{B}})$, and the fact that we consider it equal to zero allows us to replace $\bar{\mathbf{B}} \cdot (\nabla \times \bar{\mathbf{U}}) = \bar{\mathbf{U}} \cdot (\nabla \times \bar{\mathbf{B}})$.

3 IMPLEMENTING THE TWO SCALE APPROXIMATION

As was advanced in the Introduction, we shall work within the two scale approximation, whereby mean fields are supposed to peak at a scale k_L^{-1} , and stochastic ones at $k_S^{-1} \ll k_L^{-1}$. We begin by noting that eq. (12) together with the definition of the t.e.m.f. and eq. (13), are very complicated, involving new functions of the mean and stochastic fields for which further equations must be deduced. From the constraint on $\bar{\mathbf{U}}$ and $\bar{\mathbf{B}}$ derived from $\nabla \cdot \bar{\mathcal{E}} = 0$, and the fact that we are considering $\bar{\mathbf{B}}$ as force-free, we can write $\bar{\mathbf{U}} \cdot (\nabla \times \bar{\mathbf{B}}) \simeq k_L \bar{\mathbf{U}} \cdot \bar{\mathbf{B}}$, where the last semi-equality stems from the fact that $\bar{\mathbf{B}}$ is considered to be force-free. From eqs. (10) and (11), we see that the dot product of $\bar{\mathcal{E}}$ with $\bar{\mathbf{B}}$ is responsible for magnetic helicity transport. Let us write $\bar{\mathcal{E}}^{\bar{\mathbf{B}}} \equiv \bar{\mathcal{E}} \cdot \bar{\mathbf{B}}$. Its evolution equation is

² We cannot use equation (8) for \mathbf{a} to calculate $\langle \nabla \cdot \mathcal{E} \rangle_0$ as it is trivially zero.

³ Observe that in contrast to previous works, here $\bar{\mathcal{E}}^{\bar{\mathbf{B}}}$ is the dot product of the t.e.m.f. with $\bar{\mathbf{B}}$, and not the component of that force along the mean magnetic field. We chose to work with this

$\partial \bar{\mathcal{E}}^B / \partial t = (\partial \bar{\mathcal{E}} / \partial t) \cdot \bar{\mathbf{B}} + \bar{\mathcal{E}} \cdot \partial \bar{\mathbf{B}} / \partial t$. Proceeding similarly as in Refs. (Blackman & Field 2002; Kandus et al. 2006), we obtain the following full form for the evolution equation for $\bar{\mathcal{E}}^B$, in the two scale approximation:

$$\begin{aligned} \frac{\partial \bar{\mathcal{E}}^B}{\partial t} &= \frac{2}{3} k_L H_S^C H_L^C + \frac{1}{3} k_L [k_S^2 H_S^M - H^u] |H_L^M| \\ &- \frac{2}{3} k_L^2 E^u H_L^M - \zeta_1 \bar{\mathcal{E}}^B, \end{aligned} \quad (14)$$

where we replaced $\overline{\mathbf{U} \cdot \mathbf{B}} \simeq \langle \mathbf{U} \cdot \mathbf{B} \rangle_{vol} = H_L^C$, the large scale cross-helicity; $\overline{H_S^C} = \langle \mathbf{u} \cdot \mathbf{b} \rangle_{vol} \simeq \overline{\mathbf{u} \cdot \mathbf{b}}$, the small scale cross-helicity; $(\nabla \times \mathbf{b}) \cdot \mathbf{b} \simeq k_S^2 H_S^M$; $E^u = \overline{u^2}/2$ and $H^u = \mathbf{u} \cdot (\nabla \times \mathbf{u})$ where these two last quantities are considered prescribed. Since we are considering $\bar{\mathbf{B}}$ to be force-free, we replaced $|\bar{\mathbf{B}}|^2 \simeq k_L |H_L^M|$ and $(\nabla \times \bar{\mathbf{B}}) \cdot \bar{\mathbf{B}} \simeq k_L^2 H_L^M$. The last term in eq. (14), $\zeta_1 \bar{\mathcal{E}}^B$, includes the effect of viscosity, resistivity, the term $\bar{\mathcal{E}} \cdot \partial \bar{\mathbf{B}} / \partial t$ and, more importantly, the three point correlations denoted by $\bar{\mathbf{T}}$ in eq. (12). We see that besides the equations derived until now we need the ones for $H_{L,S}^C$. The derivation of these equations is sketched in the Appendix and, although being a straightforward procedure, it is a rather tedious one and we end up with a system of seven equations, besides the four ones already shown above: the two equations for $H_{L,S}^C$, the ones for $\bar{F}^B \equiv (\nabla \times \mathbf{b}) \times \mathbf{b} \cdot \bar{\mathbf{B}}$, i.e., the dot product of the small scale Lorentz force with $\bar{\mathbf{B}}$, for $\bar{F}^W \equiv (\nabla \times \mathbf{b}) \times \mathbf{b} \cdot \bar{\mathbf{W}}$, for $\bar{F}^U \equiv (\nabla \times \mathbf{b}) \times \mathbf{b} \cdot \bar{\mathbf{U}}$, for $\bar{\mathcal{E}}^W = \bar{\mathcal{E}} \cdot \bar{\mathbf{W}}$, i.e., the scalar product of the t.e.m.f. with the large scale vorticity, and for $E^b = \overline{b^2}/2$, i.e., the small scale magnetic energy. Eq. (14) is already expressed in the two scale form. The other ten equations read

$$\frac{\partial H_L^M}{\partial t} \simeq 2 \bar{\mathcal{E}}^B - 2 \eta k_L^2 H_L^M, \quad (15)$$

$$\frac{\partial H_S^M}{\partial t} \simeq -2 \bar{\mathcal{E}}^B - 2 \eta k_S^2 H_S^M, \quad (16)$$

$$\frac{\partial H^U}{\partial t} \simeq 2 \bar{F}^W - 2 \nu k_L^2 H^U, \quad (17)$$

$$\begin{aligned} \frac{\partial \bar{\mathcal{E}}^W}{\partial t} &= \frac{2}{3} k_L H_S^C |H^U| + \frac{1}{3} k_L [k_S^2 H_S^M - H^u \\ &- 2 k_L E^u] H_L^C - \zeta_2 \bar{\mathcal{E}}^W, \end{aligned} \quad (18)$$

$$\frac{\partial H_L^C}{\partial t} = \bar{F}^B + \bar{\mathcal{E}}^W - 2(\nu + \eta) k_L^2 H_L^C, \quad (19)$$

$$\frac{\partial H_S^C}{\partial t} = -\bar{F}^B - \bar{\mathcal{E}}^W - 2(\nu + \eta) H_L^C, \quad (20)$$

$$\frac{\partial \bar{F}^W}{\partial t} = \frac{1}{3} k_L^3 H_S^C H_L^C - \frac{2}{3} k_L^2 \sqrt{2 E_2^b} H^U - \zeta_F \bar{F}^W, \quad (21)$$

$$\frac{\partial \bar{F}^U}{\partial t} \simeq \frac{1}{3} k_L^2 H_S^C H_L^C - \frac{2}{3} k_L \sqrt{2 E_2^b} |H^U| - \zeta_F \bar{F}^U, \quad (22)$$

function because it facilitates the numerical integration of the equations, while preserving the generic behaviour of the projections.

$$\frac{\partial \bar{F}^B}{\partial t} \simeq \frac{1}{3} k_L^3 H_S^C |H_L^M| - \frac{2}{3} k_L^2 \sqrt{2 E_2^b} H_L^C - \zeta_F \bar{F}^B, \quad (23)$$

and

$$\frac{\partial E_2^b}{\partial t} = -2 \bar{F}^U \sqrt{E_2^b} - k_S^2 H_S^M \bar{\mathcal{E}}^B + H_S^C \bar{F}^B - \zeta_b E_2^b, \quad (24)$$

where $E_2^b = (E^b)^2$.

4 MAKING THE EQUATIONS NON-DIMENSIONAL

In order to work with non-dimensional quantities, we define the following dimensionless variables: $\tau = k_S u t$, $\mathcal{F}^W = \bar{F}^W / k_S^2 u^3$, $\mathcal{F}^U = \bar{F}^U / k_S u^3$, $\mathcal{F}^B = \bar{F}^B / k_S u^3$, $\Xi_2^b = E_2^b / u^4$, $\mathcal{Q}^B = \bar{\mathcal{E}}^B / u^3$, $\mathcal{Q}^W = \bar{\mathcal{E}}^W / k_S u^3$, $\mathcal{H}_{S,L}^M = k_S H_{S,L}^M / u^2$, $\mathcal{H}^U = H^U / k_S u^2$, $\mathcal{H}_{L,S}^C = H_{L,S}^C / u^2$, $\mathcal{H}^u = H^u / k_S u^2$, $\Xi^u = E^u u^2$, $\xi_i = \zeta_i / k_S u$ ($i = 1, 2, F, b$), $R_e = \nu k_S / u$, $R_m = \eta k_S / u$, with R_e the Reynolds number, R_m the magnetic Reynolds number, and $r = k_L / k_S$. The non-dimensional equations then read

$$\begin{aligned} \frac{\partial \mathcal{Q}^B}{\partial \tau} &= \frac{2}{3} r \mathcal{H}_S^C \mathcal{H}_L^C + \frac{1}{3} r [\mathcal{H}_S^M - \mathcal{H}^u] |\mathcal{H}_L^M| \\ &- \frac{2}{3} r^2 \Xi^u \mathcal{H}_L^M - \xi_1 \mathcal{Q}^B, \end{aligned} \quad (25)$$

$$\begin{aligned} \frac{\partial \mathcal{Q}^W}{\partial \tau} &= \frac{2}{3} r \mathcal{H}_S^C |\mathcal{H}^U| + \frac{1}{3} r [\mathcal{H}_S^M - \mathcal{H}^u - 2 r \Xi^u] \mathcal{H}_L^C \\ &- \xi_2 \mathcal{Q}^W, \end{aligned} \quad (26)$$

$$\frac{\partial \mathcal{H}_L^M}{\partial \tau} = 2 \mathcal{Q}^B - \frac{2 r^2}{R_m} \mathcal{H}_L^M, \quad (27)$$

$$\frac{\partial \mathcal{H}_S^M}{\partial \tau} = -2 \mathcal{Q}^B - \frac{2}{R_m} \mathcal{H}_S^M, \quad (28)$$

$$\frac{\partial \mathcal{H}^U}{\partial \tau} \simeq 2 \mathcal{F}^W - \frac{2 r^3}{R_e} \mathcal{H}^U, \quad (29)$$

$$\frac{\partial \mathcal{H}_L^C}{\partial \tau} = \mathcal{F}^B + \mathcal{Q}^W - \left(\frac{2}{R_m} + \frac{2}{R_e} \right) r^2 \mathcal{H}_L^C, \quad (30)$$

$$\frac{\partial \mathcal{H}_S^C}{\partial \tau} = -\mathcal{F}^B - \mathcal{Q}^W - \left(\frac{2}{R_m} + \frac{2}{R_e} \right) \mathcal{H}_S^C, \quad (31)$$

$$\frac{\partial \mathcal{F}^W}{\partial \tau} = \frac{1}{3} r^3 \mathcal{H}_S^C \mathcal{H}_L^C - \frac{2}{3} r^2 \sqrt{2 \Xi_2^b} \mathcal{H}^U - \xi_F \mathcal{F}^W, \quad (32)$$

$$\frac{\partial \mathcal{F}^U}{\partial \tau} \simeq \frac{1}{3} r^2 \mathcal{H}_S^C \mathcal{H}_L^C - \frac{2}{3} r \sqrt{2 \Xi_2^b} |\mathcal{H}^U| - \xi_F \mathcal{F}^U, \quad (33)$$

$$\frac{\partial \mathcal{F}^B}{\partial \tau} \simeq \frac{1}{3} r^3 \mathcal{H}_S^C |\mathcal{H}_L^M| - \frac{2}{3} r^2 \sqrt{2 \Xi_2^b} \mathcal{H}_L^C - \xi_F \mathcal{F}^B, \quad (34)$$

and

$$\frac{\partial \Xi_2^b}{\partial \tau} \simeq -2 \mathcal{F}^U \sqrt{\Xi_2^b} - \mathcal{H}_S^M \mathcal{Q}^B + \mathcal{H}_S^C \mathcal{F}^B - \xi_b \Xi_2^b. \quad (35)$$

5 NUMERICAL RESULTS AND DISCUSSION

We numerically integrated equations (25)-(35) using the following parameters and initial conditions: $r = 0.2$, $\mathcal{H}^u = -1$, $\Xi^u = 1$, $\mathcal{Q}^B(0) = \mathcal{Q}^W(0) = 0$, $\mathcal{H}_L^M(0) = 0.001$, $\mathcal{H}_0^M(0) = -0.001$, $\mathcal{H}^U(0) = 0.0001$, $\mathcal{H}_L^C(0) = 0.0001$,

$\mathcal{H}_S^C(0) = -0.0001$, $\mathcal{F}^B(0) = \mathcal{F}^W(0) = \mathcal{F}^U(0) = 0$, $\Xi_2^b(0) = 0$, $R_m = R_e = 200$ and 2000 (i.e., magnetic Prandtl number $P_m = 1$), $\xi_i = 1/2$ (strong non-linearities) and $\xi_i = 1/R_m$ (weak non-linearities)⁴. A comment about the chosen values for ξ_i is in order: In principle this parameter can depend on R_m ; however, results of numerical simulations show that it is of order unity for $R_m < 100$. In this sense, the value $\xi = 1/2$ would be in accord with those results. As we are working here with larger values of R_m for which, to our knowledge, there lacks numerical estimations of ξ_i , we chose two values that might represent the two extreme behaviours of this parameter. Nevertheless, we must stress that the validity of this choice should be checked by direct numerical simulations. In Fig. 1 we plotted \mathcal{H}^U as a function of τ for $\xi_i = 1/2$. The long dashed line corresponds to $R_m = 200$ while the short dashed one to $R_m = 2000$. We see that the generation of \overline{U} is rather weak, but the effect seems to be stronger for $R_m = 2000$ as time passes. In Fig. 2 we plotted $\mathcal{H}^{\overline{U}}$ as a function of τ for $\xi_i = 1/R_m$. The full line corresponds to $R_m = 200$ while the dotted one to $R_m = 2000$. In this case there is a strong production of large scale kinetic helicity, it being stronger for $R_m = 2000$ at the beginning of the integration, while for later times there seems to be no difference between the outcomes for the two R_m considered. In Fig. 3 we plotted the logarithm of the small scale magnetic energy, $\ln(\Xi_2^b)$ as a function of τ , for $\xi_i = 1/2$. Each curve consists of two curves: one with the effect of \overline{U} and the other without this field. This superposition of curves means that for the chosen value of ξ_i the effect of \overline{U} on the evolution of small scale magnetic energy is negligible. The upper curve corresponds to the largest value of R_m , and we see that in this case a saturation value for Ξ_2^b larger than for $R_m = 200$ is attained. In Fig. 4 we plotted the logarithm of the small scale magnetic energy $\ln(\Xi_b)$, for $\xi_i = 1/R_m$. Long dash curves correspond to $R_m = 2000$: upper curve contains the effect of \overline{U} , lower oscillating curve is without the action of those fields. Short dashed curves correspond to $R_m = 200$, with the same features for the presence and absence of \overline{U} . We see that the action of \overline{U} strongly enhances the generation of small scale magnetic energy, and again this effect is stronger for larger R_m . In Fig. 5 we plotted Q^B as a function of τ for $\xi_i = 1/2$. Here again each curve consists of two curves, one with the effect of \overline{U} and the other without, showing again that for strong non-linearities the effect of those flows is negligible, consistently with previous figures. Upper curve corresponds to $R_m = 2000$ while lower curve to $R_m = 200$. We see that again for larger R_m there is an enhancement of Q^B . In Fig. 6 we plotted Q^B as a function of τ for $\xi_i = 1/R_m$. We see here again that the action of large scale flows enhances the mean electromotive force Q^B and this enhancement is stronger for larger R_m . Oscillating, dotted-line curve corresponds to $R_m = 2000$, and full-line curve to $R_m = 200$. The curves corresponding to the absence of the effect of those flows are almost indistinguishable from the τ axis. In Fig. 7 we plotted \mathcal{H}_L^M as a function of τ for $\xi_i = 1/2$. Here again each curve consists of two curves, one with the effect of \overline{U} and the other without, showing again that for strong non-linearities the effect of

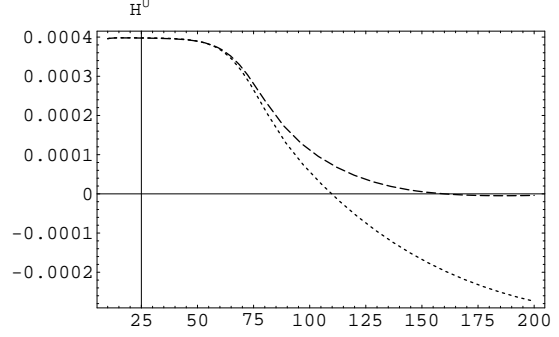


Figure 1. Large scale kinetic helicity \mathcal{H}^u as a function of τ for $\xi_i = 1/2$. The long dashed line corresponds to $R_m = 200$, and the dotted line to $R_m = 2000$. The generation of $H^{\overline{U}}$ is stronger for the largest value of R_m .

those flows is negligible. Fast growing curve corresponds to $R_m = 200$ while lower one to $R_m = 2000$. The coincidence of the two curves for short times corresponds to the kinematic regime, where back-reaction of the induced magnetic fields \mathbf{b} did not take place yet. In Fig. 8 we plotted \mathcal{H}_L^M as a function of τ for $\xi_i = 1/R_m$. We see here again that the action of large scale flows enhances the mean electromotive force \mathcal{H}_L^M and this enhancement is stronger for larger R_m . Dashed curves correspond to $R_m = 2000$: the ones with the largest amplitude correspond to the action of \overline{U} , while the lower amplitude to the absence of this effect. Full line correspond to $R_m = 200$, and the features with respect to the presence and absence of large scale flows are the same as for $R_m = 200$. The coincidence of all four curves at the beginning of the evolution corresponds to the kinematic regime. In Fig. 9 we plotted \mathcal{H}_L^C as a function of τ for $\xi_i = 1/2$. Dashed line curve corresponds to $R_m = 2000$ while full line to $R_m = 200$. Consistently with Fig. (1), we see that \mathcal{H}_L^C is larger for larger R_m . In Fig. 10 we plotted \mathcal{H}_L^C as a function of τ for $\xi_i = 1/R_m$. Dotted line corresponds to $R_m = 2000$ while full line to $R_m = 200$. Consistently with Fig. (2), we see that \mathcal{H}_L^C is larger for $R_m = 2000$ than for $R_m = 200$, with the difference in amplitudes between both quantities getting smaller with time.

6 CONCLUSIONS

In this paper we studied semi-analytically and qualitatively the generation of large scale flows by the action of a turbulent mean field dynamo, and the back-reaction of those flows on the turbulent electromotive force for two values of magnetic Reynolds number, $R_m = 200$ and 2000 , and magnetic Prandtl number $P_m = 1$. We considered a system in which small scale turbulent flows are fully helical and prescribed by a given external mechanism, i.e., a kinematically driven dynamo, and that this system possesses boundary conditions such that all total divergencies vanish. The turbulence was considered to be homogeneous and isotropic, which although being of limited applicability to obtain quantitative results for real systems, it serves to study many conceptual aspects of large scale magnetic field generation, besides enormously simplifying the mathematics. We followed the evo-

⁴ As before, subindex "i" denotes 1, 2, F and b. In each integration we assume that all ξ_i are the same for all "i".

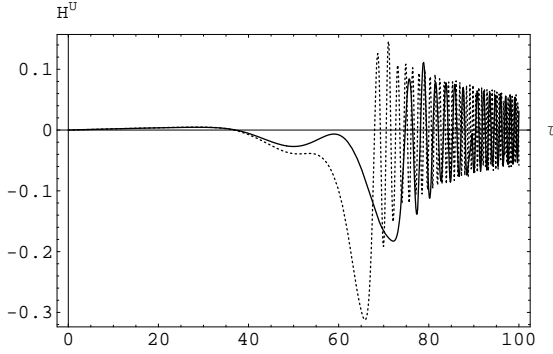


Figure 2. Large scale kinetic helicity \mathcal{H}^u as a function of τ for $\xi_i = 1/R_m$. The continuum line corresponds to $R_m = 200$, and the dotted one to $R_m = 200$. At the beginning, the induction of H^U is stronger for the largest value of R_m .

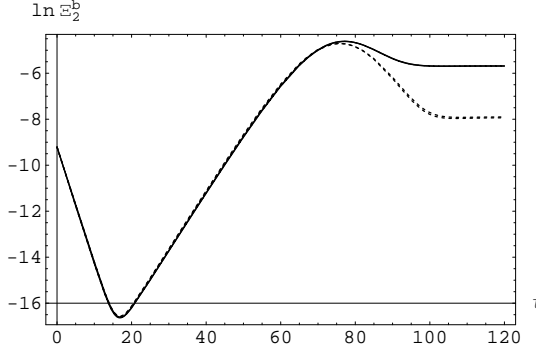


Figure 3. Logarithm of the small scale magnetic energy Ξ_2^b as a function of τ , for $\xi_i = 1/2$. Each curve is in fact two curves, with and without the effect of \overline{U} , which means that in this case the effect of those flows is negligible. The upper curve corresponds to $R_m = 2000$ and the lower one to $R_m = 200$. The small scale magnetic energy density is very small although larger for $R_m = 2000$

lution of large scale flows through their associated kinetic helicity H^U , as one of the suppositions we made was that those flows were fully helical too. One crucial assumption we made was that $\nabla \cdot \overline{\mathbf{E}} = 0$, which due to the Coulomb gauge, imposed a further constraint on $\overline{\mathbf{U}}$ and $\overline{\mathbf{B}}$ that allowed us to expressed several terms as large scale cross-helicity. Another one was to assume that $\overline{\mathbf{B}}$ is force-free and the main reason to adopt it is that it helped to simplify the mathematics. Although this condition can be fulfilled in certain astrophysical environments, this is not a general situation, therefore it should be dropped off in future works that aim at generalizing this work.

We found that large scale flows act on the t.e.m.f. $\overline{\mathbf{E}}$ through large and small scale cross-helicities, and that, for the minimal τ closure considered here, the effect of those fields is stronger for large relaxation times ($\xi_i = 1/R_m$). For short relaxation time ($\xi_i = 1/2$), the effect of those fields seems to be negligible. The choice of the values for ξ_i was arbitrary, in the sense that, to our present knowledge, it

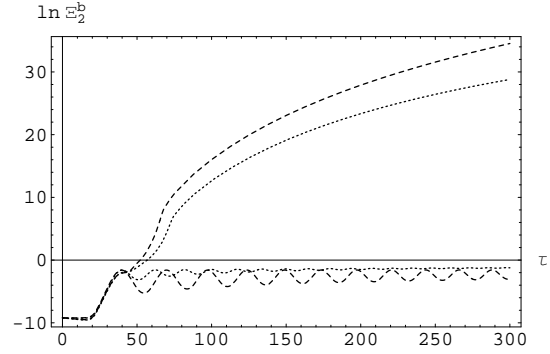


Figure 4. Logarithm of the small scale magnetic energy Ξ_2^b as a function of τ for $\xi_i = 1/R_m$. The upper, long dash curve corresponds to the presence of \overline{U} , while the lower, oscillating one, to the absence of those flows, both of them for $R_m = 2000$. Short dashed curves represents the same quantities but for $R_m = 200$. In this case large scale flows strongly enhance the production of small scale magnetic energy, and this effect is again stronger for larger values of R_m .

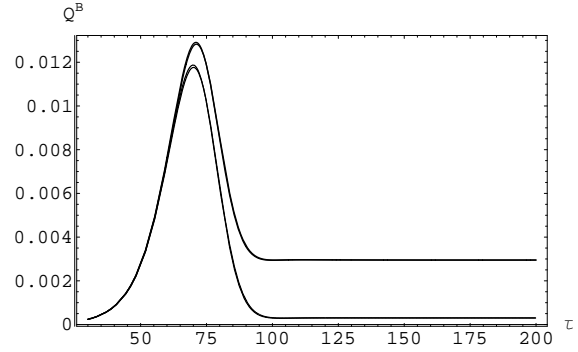


Figure 5. Mean electromotive force Q^B as function of τ for $\xi_i = 1/2$. Each curve is two curves, one with the effect of \overline{U} and the other without those fields, showing that for this value of ξ the effect of those fields is negligible. Upper curve corresponds to $R_m = 2000$ and lower curve to $R_m = 200$, which shows that for the largest R_m , Q^B is slightly stronger.

is not known how that parameter depends on the magnetic Reynolds number for $R_m > 100$. For $R_m < 100$ it seems to be confirmed that ξ_i is of order unity. Here we chose to work with two values that may be considered as representative of two extreme possibilities: $\xi_i = 1/2$ would be consistent with the predictions of the numerical simulations (although they were made for a different R_m interval), while $\xi = 1/R_m$ would represent a resistive case. In any case, more reliable values should be given by numerical simulations performed for $R_m > 100$.

Due to the simple system considered and the approximations we made, we do not intended to find quantitative results, as for example estimate the time interval during which $\overline{\mathbf{U}} \times \overline{\mathbf{B}} \ll \overline{\mathbf{E}}$ is valid, nor do we extract more conceptual and qualitative conclusions. We end this work stressing the im-

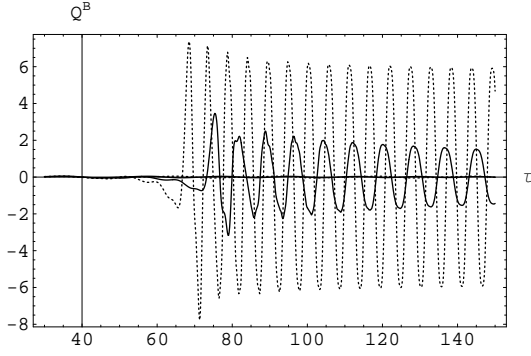


Figure 6. Mean electromotive force Q^B as function of τ for $\xi_i = 1/R_m$. Dotted line corresponds to $R_m = 2000$ and continuous line to $R_m = 200$. The curves corresponding to the absence of large scale flows are of negligible amplitude and almost indistinguishable from the τ axis, showing that in this case the enhancement of the t.e.m.f. by the shear fields is very strong.

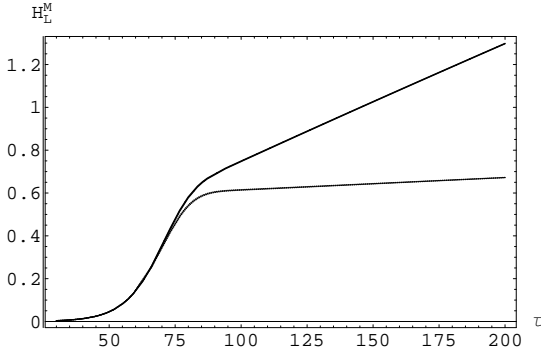


Figure 7. Large scale magnetic helicity \mathcal{H}_L^M as a function of τ , for $\xi_i = 1/2$. Again in this figure, each curve is two curves, one with the effect of \overline{U} and the other without, showing that the effect of those flows on the evolution of magnetic helicity is negligible for the chosen value of ξ_i . The growing curve correspond to $R_m = 200$, while the slowly growing, lower curve to $R_m = 2000$.

portance of studying this problem via numerical simulations, that will show us the next paths to follow in a further analytical study, besides confirming or contesting the results presented here. The semi-analytical study of the anisotropic case is also of the most importance, as well as the consideration of other boundary conditions.

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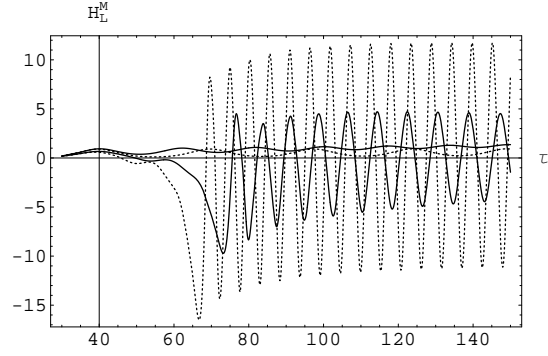


Figure 8. Large scale magnetic helicity \mathcal{H}_L^M as a function of τ , for $\xi_i = 1/R_m$. Dotted curves correspond to $R_m = 2000$ while continuous curves to $R_m = 200$. In each case, the strongly oscillating curves correspond to the action of large scale flows, while the slowly oscillations to their absence. Consistently with what was shown in Fig. 6, the action of \overline{U} enhances the cascade of magnetic helicity.

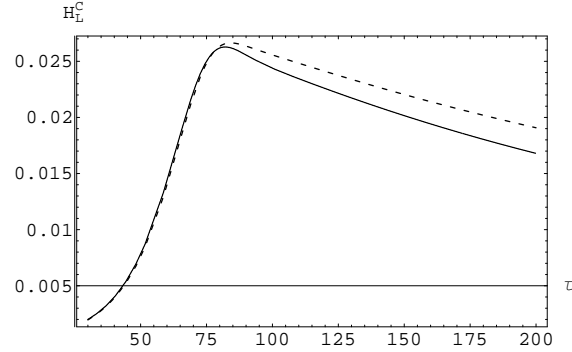


Figure 9. Large scale cross helicity \mathcal{H}_L^C as function of τ for $\xi_i = 1/2$. Consistently with Fig. 1 the generation of large scale cross helicity is stronger for $R_m = 2000$ (dashed line) than for $R_m = 200$ (full line).

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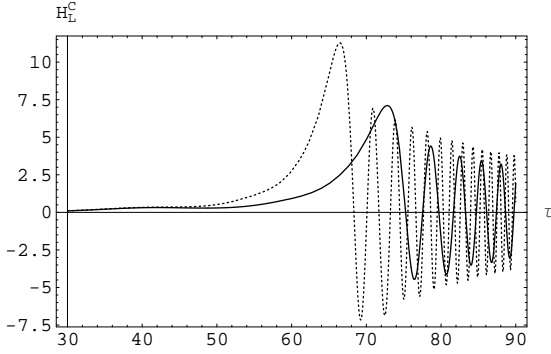


Figure 10. Large scale cross helicity \mathcal{H}_L^C as function of τ , for $\xi_i = 1/R_m$. Consistently with Fig. 2 the generation of large scale cross-helicity is stronger for $R_m = 2000$ (dotted line) than for $R_m = 200$ (continuous line).

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APPENDIX A: DEDUCTION OF THE COMPLEMENTARY EVOLUTION EQUATIONS

Here we sketch the derivation of the evolution equation for the large scale kinetic helicity as well as the set of extra equations needed to study the problem considered in this article.

A1 Evolution Equation for the Large Scale Vorticity

We start from Navier-Stokes equation written in the form

$$\begin{aligned} \frac{\partial \mathbf{U}}{\partial t} = & -(\nabla \times \mathbf{U}) \times \mathbf{U} - \nabla \left(\frac{U^2}{2} + \frac{p}{\rho} \right) \\ & + (\nabla \times \mathbf{B}) \times \mathbf{B} + \nu \nabla^2 \mathbf{U}. \end{aligned} \quad (\text{A1})$$

The equation for $\overline{\mathbf{W}} = \nabla \times \overline{\mathbf{U}}$ is obtained by simply taking the curl of eq. (A1), after replacing the decomposition in mean and stochastic fields, and using the hypothesis that \mathbf{u} is fully helical. We have

$$\begin{aligned} \frac{\partial \overline{\mathbf{W}}}{\partial t} = & -\nabla \times (\overline{\mathbf{W}} \times \overline{\mathbf{U}}) + \nabla \times \overline{(\nabla \times \mathbf{b}) \times \mathbf{b}} \\ & + \nu \nabla^2 \overline{\mathbf{W}}. \end{aligned} \quad (\text{A2})$$

A2 Evolution Equation for the Large Scale Kinetic Helicity

It is obtained as $\partial H^{\overline{\mathbf{U}}}/\partial t = \langle \overline{\mathbf{U}} \cdot \partial \overline{\mathbf{W}}/\partial t \rangle_{vol} + \langle \overline{\mathbf{W}} \cdot \partial \overline{\mathbf{U}}/\partial t \rangle_{vol}$. Replacing the corresponding equations we obtain

$$\frac{\partial H^{\overline{\mathbf{U}}}}{\partial t} \simeq 2 \overline{(\nabla \times \mathbf{b}) \times \mathbf{b} \cdot \overline{\mathbf{W}}} - 2\nu k_L^2 H^{\overline{\mathbf{U}}}, \quad (\text{A3})$$

where the semi-equality stems from the fact that we approximated $\langle \dots \rangle_{vol} \simeq \dots$. We define $\overline{\mathbf{F}} \equiv \overline{(\nabla \times \mathbf{b}) \times \mathbf{b} \cdot \overline{\mathbf{W}}}$, and thus eq. (A3) reads

$$\frac{\partial H^{\overline{\mathbf{U}}}}{\partial t} = 2\overline{\mathbf{F}} - 2\nu k_L^2 H^{\overline{\mathbf{U}}}. \quad (\text{A4})$$

A3 Evolution Equation for $\overline{\mathbf{F}} \equiv \overline{(\nabla \times \mathbf{b}) \times \mathbf{b} \cdot \overline{\mathbf{W}}}$ and its Projections

It is found by taking curl of eq. (7) and using it to expand $[\partial(\nabla \times \mathbf{b})/\partial t] \times \mathbf{b} + (\nabla \times \mathbf{b}) \times \partial \mathbf{b}/\partial t$. After a somewhat lengthy, but straightforward calculation, where it was assumed that for the two scale approximation $\nabla \cdot \overline{\mathbf{F}} \simeq 0^5$, we obtain

$$\begin{aligned} \frac{\partial \overline{\mathbf{F}}}{\partial t} \simeq & \frac{1}{3} \overline{(\nabla \cdot \overline{\mathcal{E}}) (\nabla \times \overline{\mathbf{B}})} - \frac{k_S}{3} \overline{(\nabla \cdot \overline{\mathcal{E}}) \overline{\mathbf{B}}} \\ & - \frac{1}{3} H_S^C \nabla^2 \overline{\mathbf{B}} + \frac{2}{3} E^b \nabla^2 \overline{\mathbf{U}} - \zeta_F \overline{\mathbf{F}}. \end{aligned} \quad (\text{A5})$$

As in the body of the paper, we assume that $\overline{\nabla \cdot \mathcal{E}} = 0$, so eq. (A5) reduces to

$$\frac{\partial \overline{\mathbf{F}}}{\partial t} \simeq -\frac{1}{3} H_S^C \nabla^2 \overline{\mathbf{B}} + \frac{2}{3} E^b \nabla^2 \overline{\mathbf{U}} - \zeta_F \overline{\mathbf{F}}. \quad (\text{A6})$$

To find the evolution equations for the scalar product of $\overline{\mathbf{F}}$ with $\overline{\mathbf{W}}$ and $\overline{\mathbf{B}}$, we use the above defined expression $\overline{\mathbf{F}}^{\overline{\mathbf{W}}}$, and an analogous expression for $\overline{\mathbf{B}}$. Again the evolution equation is found by taking the time derivative of the complete expression. In the two scale approximation we have

$$\begin{aligned} \frac{\partial \overline{\mathbf{F}}^{\overline{\mathbf{W}}}}{\partial t} \simeq & -\frac{1}{3} H_S^C \nabla^2 \overline{\mathbf{B}} \cdot \overline{\mathbf{W}} + \frac{2}{3} E^b \nabla^2 \overline{\mathbf{U}} \cdot \overline{\mathbf{W}} - \zeta_F \overline{\mathbf{F}}^{\overline{\mathbf{W}}} \\ = & \frac{k_L^2}{3} H_S^C \overline{\mathbf{B}} \cdot \overline{\mathbf{W}} - \frac{2k_L^2}{3} E^b H^{\overline{\mathbf{U}}} - \zeta_F \overline{\mathbf{F}}^{\overline{\mathbf{W}}}. \end{aligned} \quad (\text{A7})$$

Due to the fact that $\nabla \cdot \overline{\mathcal{E}} = 0 \Rightarrow \nabla \cdot (\overline{\mathbf{U}} \times \overline{\mathbf{B}}) \simeq 0$, we can write $\overline{\mathbf{B}} \cdot \overline{\mathbf{W}} \simeq \overline{\mathbf{U}} \cdot (\nabla \times \overline{\mathbf{B}}) \simeq k_L \overline{\mathbf{U}} \cdot \overline{\mathbf{B}} \simeq k_L H_L^C$. Using the fact that for a fully helical $\overline{\mathbf{U}}$ field we can write $|\overline{\mathbf{W}}| \simeq k_L^{1/2} |\overline{H^{\mathbf{U}}}|^{1/2}$, we obtain

$$\frac{\partial \overline{\mathbf{F}}^{\overline{\mathbf{W}}}}{\partial t} \simeq \frac{k_L^3}{3} H_S^C H_L^C - \frac{2k_L^2}{3} E^b H^{\overline{\mathbf{U}}} - \zeta_F \overline{\mathbf{F}}^{\overline{\mathbf{W}}}. \quad (\text{A8})$$

For the projection of $\overline{\mathbf{F}}$ along $\overline{\mathbf{B}}$ and along $\overline{\mathbf{U}}$ we proceed analogously as for $\overline{\mathbf{F}}^{\overline{\mathbf{W}}}$. Using the fact that for a large scale force-free field we can write $|\mathbf{B}_0| = k_L^{1/2} |H_L^M|^{1/2}$, we obtain

⁵ $\nabla \cdot \langle (\nabla \times \mathbf{b}) \times \mathbf{b} \rangle = -\mathbf{b} \cdot \nabla^2 \mathbf{b} - |\nabla \times \mathbf{b}|^2 \simeq k_S^2 |\mathbf{b}|^2 - k_S^2 |\mathbf{b}|^2 = 0$

$$\frac{\partial \overline{F^B}}{\partial t} \simeq \frac{k_L^3}{3} H_S^C |H_L^M| - \frac{2k_L^2}{3} E^b H_L^C - \zeta_F \overline{F^B}, \quad (\text{A9})$$

and

$$\frac{\partial \overline{F^U}}{\partial t} \simeq \frac{k_L^2}{3} H_S^C H_L^C - \frac{2k_L}{3} E^b |H^U| - \zeta_F \overline{F^U} \quad (\text{A10})$$

where we used $|\overline{\mathbf{U}}|^2 \simeq |H^U|/k_L$.

A4 Evolution Equation for the Cross-Helicity

Cross-Helicity is defined as $H^C = \langle \mathbf{U} \cdot \mathbf{B} \rangle_{vol}$. After obtaining from eq. (A1) the evolution equations for $\overline{\mathbf{U}}$ and \mathbf{u} and using eq. (4) and (7), we obtain the following equation for the large scale cross helicity, H_L^C and the small scale cross helicity H_S^C :

$$\begin{aligned} \frac{\partial H_L^C}{\partial t} &\simeq \overline{(\nabla \times \mathbf{b}) \times \mathbf{b} \cdot \overline{\mathbf{B}}} + \overline{\mathcal{E}} \cdot \overline{\mathbf{W}} \\ &- 2(\nu + \eta) k_L^2 H_L^C \end{aligned} \quad (\text{A11})$$

and

$$\begin{aligned} \frac{\partial H_S^C}{\partial t} &\simeq -\overline{(\nabla \times \mathbf{b}) \times \mathbf{b} \cdot \overline{\mathbf{B}}} - \overline{\mathcal{E}} \cdot \overline{\mathbf{W}} \\ &- 2(\nu + \eta) k_S^2 H_S^C. \end{aligned} \quad (\text{A12})$$

Replacing $\overline{F^B} \equiv \overline{(\nabla \times \mathbf{b}) \times \mathbf{b} \cdot \overline{\mathbf{B}}}$, defining $\overline{\mathcal{E}^W} \equiv \overline{\mathcal{E}} \cdot \overline{\mathbf{W}}$, and using the fact that for fully helical $\overline{\mathbf{U}}$ we can write $|\overline{\mathbf{W}}| \simeq k_L^{1/2} |H^U|^{1/2}$, in the two scale approximation we have

$$\frac{\partial H_L^C}{\partial t} \simeq \overline{\mathcal{E}^W} + \overline{F^B} - 2(\nu + \eta) k_L^2 H_L^C \quad (\text{A13})$$

and

$$\frac{\partial H_S^C}{\partial t} \simeq -\overline{\mathcal{E}^W} - \overline{F^B} - 2(\nu + \eta) k_S^2 H_S^C. \quad (\text{A14})$$

A5 Evolution Equation for $\overline{\mathcal{E}^W} = \overline{\mathcal{E}} \cdot \overline{\mathbf{W}}$

Using equation (12), and $\nabla \cdot \overline{\mathcal{E}} = 0$, we obtain

$$\begin{aligned} \frac{\partial \overline{\mathcal{E}^W}}{\partial t} &\simeq \frac{2}{3} H_S^C |\overline{\mathbf{W}}|^2 + \frac{k_L}{3} (k_S^2 H_S^M - H^u) H_L^C \\ &- \frac{1}{3} u^2 (\nabla \times \overline{\mathbf{B}}) \cdot \overline{\mathbf{W}} - \zeta_2 \overline{\mathcal{E}^W}, \end{aligned} \quad (\text{A15})$$

where in the last term we considered the term $\overline{\mathcal{E}} \cdot \partial \overline{\mathbf{W}} / \partial t$, and the three point correlations. Performing $(\nabla \times \overline{\mathbf{B}}) \cdot \overline{\mathbf{W}} \simeq k_L \overline{\mathbf{B}} \cdot \overline{\mathbf{W}} \simeq k_L (\nabla \times \overline{\mathbf{B}}) \cdot \overline{\mathbf{U}} \simeq k_L^2 H_L^C$, where the semi-equality before the last stems from the fact that $\nabla \cdot \overline{\mathcal{E}} = -\nabla \cdot (\overline{\mathbf{U}} \times \overline{\mathbf{B}}) \simeq 0$, we obtain

$$\begin{aligned} \frac{\partial \overline{\mathcal{E}^W}}{\partial t} &\simeq \frac{2}{3} k_L H_S^C |H^{U_0}| + \frac{1}{3} k_L [k_S^2 H_S^M - H^u - 2k_L E^u] H_L^C \\ &- \zeta_2 \overline{\mathcal{E}^W}. \end{aligned} \quad (\text{A16})$$

A6 Evolution Equation for E^b

It is obtained by scalar multiplying eq. (7) by \mathbf{b} and then taking volume average. In order to simplify the mathematics, we approximate the volume averages by a dot product

between spatial averages of functions of stochastic and mean fields.

$$\frac{\partial E^b}{\partial t} \simeq -\overline{(\nabla \times \mathbf{b}) \times \mathbf{b} \cdot \overline{\mathbf{U}}} + \overline{(\nabla \times \mathbf{b}) \times \mathbf{u} \cdot \overline{\mathbf{B}}} - \zeta_b E^b. \quad (\text{A17})$$

To deal with the second term we write $\nabla \times \mathbf{b} = [(\nabla \times \mathbf{b}) \cdot \mathbf{b}] \mathbf{b} / |\mathbf{b}|^2 - [(\nabla \times \mathbf{b}) \times \mathbf{b}] \times \mathbf{b} / |\mathbf{b}|^2$ and thus

$$\begin{aligned} \overline{(\nabla \times \mathbf{b}) \times \mathbf{u}} &= \frac{\overline{[(\nabla \times \mathbf{b}) \cdot \mathbf{b}] \mathbf{b} \times \mathbf{u}}}{|\mathbf{b}|^2} \\ &- \frac{\overline{\{[(\nabla \times \mathbf{b}) \times \mathbf{b}] \times \mathbf{u}\} \times \mathbf{u}}}{|\mathbf{b}|^2} \\ &\simeq -\frac{\overline{(\nabla \times \mathbf{b}) \cdot \mathbf{b}} \overline{\mathcal{E}}}{|\mathbf{b}|^2} - \frac{\overline{\{[(\nabla \times \mathbf{b}) \times \mathbf{b}] \cdot \mathbf{u}\} \mathbf{b}}}{|\mathbf{b}|^2} \\ &+ \frac{\overline{(\nabla \times \mathbf{b}) \times \mathbf{b} \mathbf{u} \cdot \mathbf{b}}}{|\mathbf{b}|^2}. \end{aligned} \quad (\text{A18})$$

We obtain for the second term in eq. (A17):

$$\begin{aligned} \overline{(\nabla \times \mathbf{b}) \times \mathbf{u} \cdot \overline{\mathbf{B}}} &\simeq -\frac{\overline{(\nabla \times \mathbf{b}) \cdot \mathbf{b}} \overline{\mathcal{E}} \cdot \overline{\mathbf{B}}}{|\mathbf{b}|^2} \\ &+ \frac{\overline{\mathbf{u} \cdot \mathbf{b} (\nabla \times \mathbf{b}) \times \mathbf{b} \cdot \overline{\mathbf{B}}}}{|\mathbf{b}|^2} \\ &\simeq -\frac{k_S^2 H_S^M \overline{\mathcal{E}^B}}{2E^b} + \frac{H_S^C \overline{F^B}}{2E^b}, \end{aligned} \quad (\text{A19})$$

where the second term in the second row of expr. (A18) was considered to give a null contribution when averaged. Defining $E_2^b \equiv (E^b)^2$ we can write the evolution equation for the small scale magnetic energy as

$$\frac{\partial E_2^b}{\partial t} \simeq -2\overline{F^U} \sqrt{E_2^b} - k_S^2 H_S^M \overline{\mathcal{E}^B} + H_S^C \overline{F^B} - \zeta_b E_2^b. \quad (\text{A20})$$